Math 250

5.1 Approximating Area Under a Curve

Briggs

Objectives:

1) Approximate the area of a plane region
   (the space bounded by the x-axis, the graph of a function, and two vertical lines)

   using
   - a) left Riemann sums
   - b) right Riemann sums
   - c) midpoint Riemann sums

   on a uniform ("regular") partition.

Building-block skills needed:

- notation describing a partition
- how to calculate partition
- sigma/summation notation.

<u>Note</u>: the sum formulas on p. 287 will not be covered on HW, PQ, or exam.

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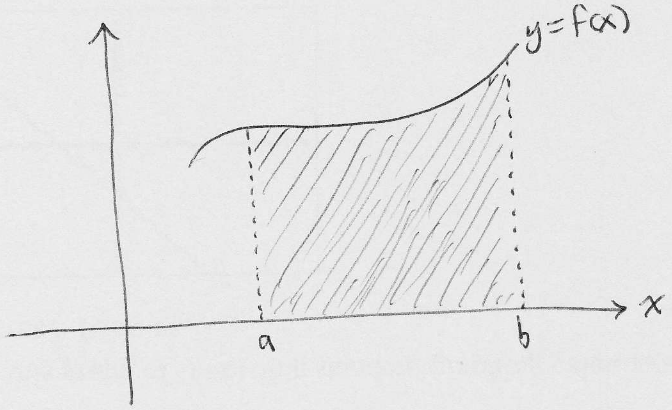
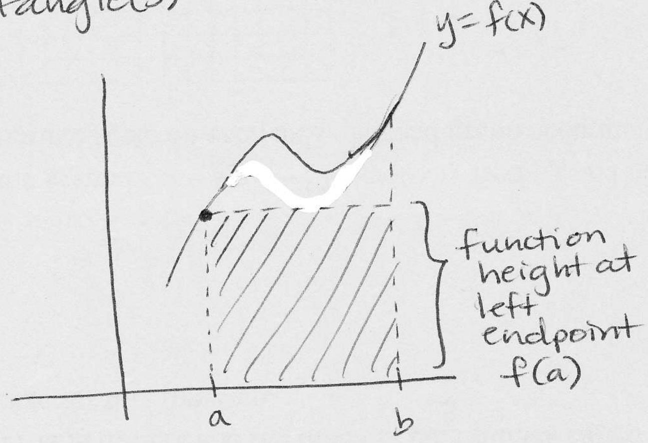
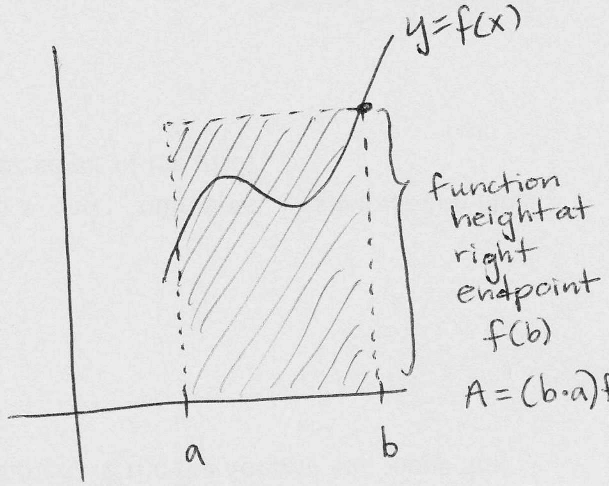
The area of a plane region bounded by $y=f(x)$, the x-axis, $x=a$, and $x=b$ is this shaded region.



For now we will concern ourselves only with functions whose graphs are entirely in QI.

A larger calculus goal is to calculate the area of the plane region exactly. For now, we begin by approximating the area of the plane region, by finding the area(s) of rectangle(s)

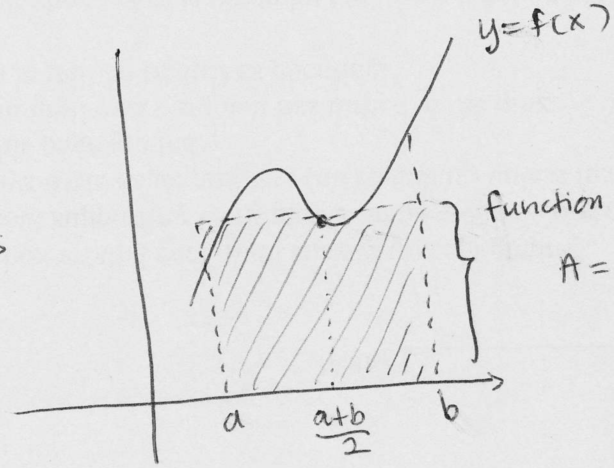




$A=(b\cdot a)f(b)$

$A=(b-a)\cdot f(a)$.

Depending on the function, each of these approximations can be bad or very bad.



function height at midpoint $f\left(\frac{a+b}{2}\right)$

$A=(b-a)\cdot f\left(\frac{a+b}{2}\right)$

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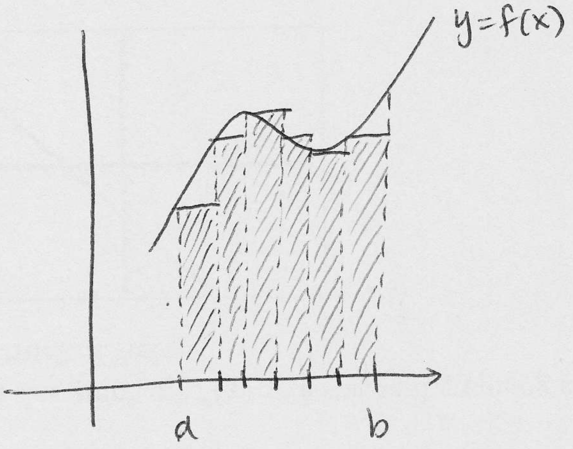
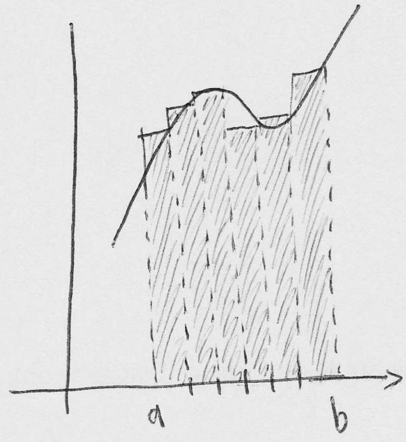
To make any of the approximations more accurate, we want to divide the interval $[a, b]$ into more rectangles – subintervals.



function height at left endpoint of each subinterval.



function height at right endpoint of each subinterval.

The more subintervals, the more accurate the approximation.

(we could also do the midpoint of each subinterval, but I've spared you the bad artwork.)

The sum of these areas:

$$\sum_{i=1}^{n} \Delta x_i \cdot f(x_i)$$ is

called a <u>Riemann Sum</u>.
"REE-mahn" (German)

First, we need to divide the interval into equal subintervals. This division is called a <u>regular partition</u> or a <u>uniform partition</u>.

n = the number of subintervals to be used.
(This will be given in the question.)

Δx = the width of each subinterval — since the partition is regular or uniform, all subintervals will be the same.

x_i = the x-coordinates, equally spaced between $x=a$ and $x=b$

Formulas: $\Delta x = \dfrac{b-a}{n}$

$x_0 = a$
$x_1 = a + \Delta x$
$x_2 = x_1 + \Delta x$ etc.... $x_n = b$.

① Find a uniform partition of $[2,10]$ with $n=6$.

$$\Delta x = \frac{b-a}{n} = \frac{10-2}{6} = \frac{8}{6} = \frac{4}{3}$$

$x_0 = a = 2$

$x_1 = 2 + \frac{4}{3} = \frac{10}{3} = 3\frac{1}{3}$

$x_2 = \frac{10}{3} + \frac{4}{3} = \frac{14}{3} = 4\frac{2}{3}$

$x_3 = \frac{14}{3} + \frac{4}{3} = \frac{18}{3} = 6$

$x_4 = \frac{18}{3} + \frac{4}{3} = \frac{22}{3} = 7\frac{1}{3}$

$x_5 = \frac{22}{3} + \frac{4}{3} = \frac{26}{3} = 8\frac{2}{3}$

$x_6 = \frac{26}{3} + \frac{4}{3} = \frac{30}{3} = 10$

1st subinterval $[2, \frac{10}{3}]$

2nd subinterval $[\frac{10}{3}, \frac{14}{3}]$

3rd $[\frac{14}{3}, 6]$

4th $[6, \frac{22}{3}]$

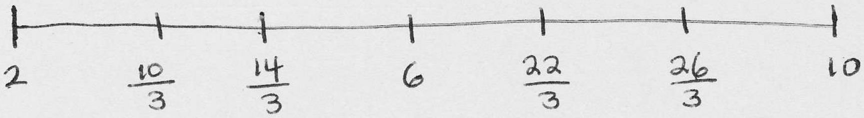
5th $[\frac{22}{3}, \frac{26}{3}]$

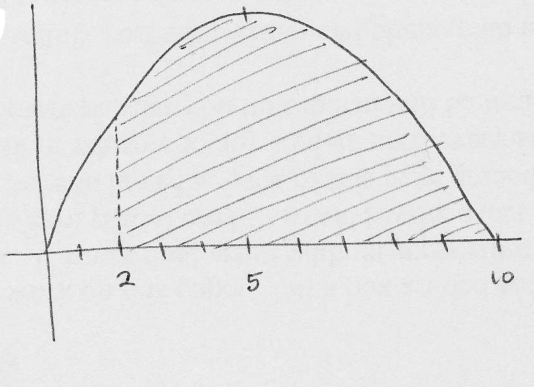
6th $[\frac{26}{3}, 10]$

② Approximate the area of the plane region bounded by $f(x) = -\frac{1}{2}x^2 + 5x$ on $[2,10]$ using $n=6$ and

a) left Riemann sum
b) right Riemann sum
c) midpoint Riemann sum

Our partion was found in the previous question.





area to be approximated

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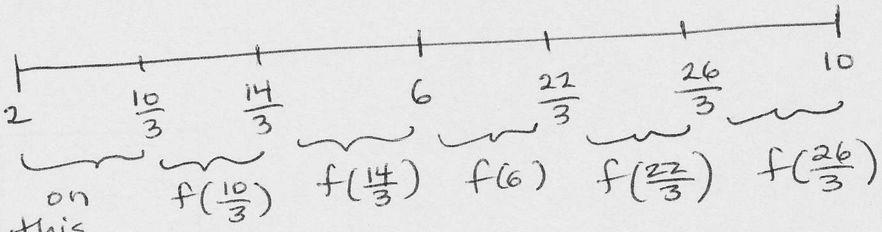
a) Left Riemann sum



Note: $f(10)$ is not used for left sum.

on this subinterval use left endpt for function height

$f(2)$

Each rectangle's width is $\Delta x = \frac{4}{3}$

$\sum_{i=1}^{n} \Delta x f(x_i^*)$ $\quad * =$ select left/right midpoint

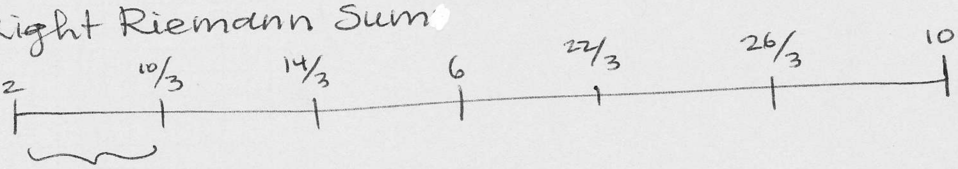
same for all, factor out

$$\text{Area} \approx \frac{4}{3}\left[f(2) + f\left(\frac{10}{3}\right) + f\left(\frac{14}{3}\right) + f(6) + f\left(\frac{22}{3}\right) + f\left(\frac{26}{3}\right)\right]$$

$$= \boxed{\frac{2128}{27}} = \boxed{78.\overline{814}}$$

Hint: use $\boxed{\text{VARS}}$ on GC!

b) Right Riemann Sum



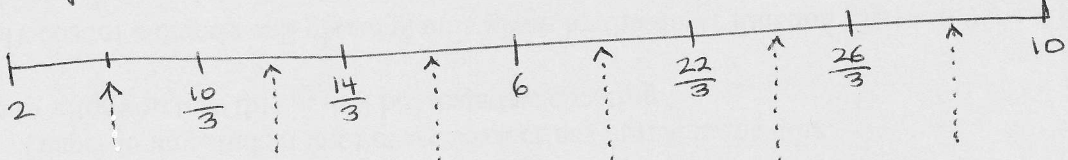
right endpoint

Note: $f(2)$ is not used for right sum

$$\text{Area} \approx \frac{4}{3}\left[f\left(\frac{10}{3}\right) + f\left(\frac{14}{3}\right) + f(6) + f\left(\frac{22}{3}\right) + f\left(\frac{26}{3}\right) + f(10)\right]$$

$$= \boxed{\frac{1840}{27}} = \boxed{68.\overline{148}}$$

c) Midpoint Riemann Sum



midpoint $= \frac{\left(2+\frac{10}{3}\right)}{2} = \frac{8}{3}$ $\quad \frac{12}{3} = 4$ $\quad \frac{16}{3}$ $\quad \frac{20}{3}$ $\quad \frac{24}{3} = 8$ $\quad \frac{28}{3}$

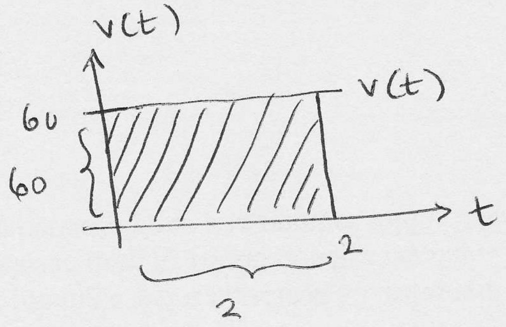
each rectangle still $\frac{4}{3}$ units wid

$$\text{Area} \approx \frac{4}{3}\left[f\left(\frac{8}{3}\right) + f(4) + f\left(\frac{16}{3}\right) + f\left(\frac{20}{3}\right) + f(8) + f\left(\frac{28}{3}\right)\right] = \boxed{\frac{2032}{27} = 75.25}$$

② Suppose the velocity of a car is 60 mph (constant) for 2 hours. What is the displacement of the car?

$$\frac{60 \text{ mi}}{\text{hr}} \times 2 \text{ hr} = 120 \text{ mi}$$

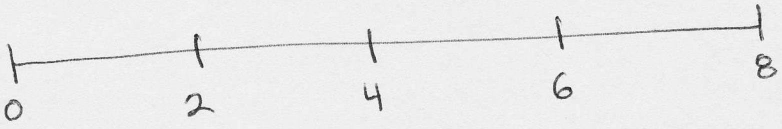
Graphically



120 mi = area of plane region under $v(t)$ on time interval $[0, 2]$.

③ Estimate the displacement if the velocity of the object is $v = \frac{1}{2t+1}$ m/s for $0 \le t \le 8$, $n = 4$.

using left endpoint of each subinterval.



$$\Delta x = \frac{8-0}{4} = 2$$

$x_0 = 0 \quad x_1 = 2 \quad x_2 = 4 \quad x_3 = 6 \quad x_4 = 8$

$$\text{displacement} \approx \Delta x \left[f(x_0) + f(x_1) + f(x_2) + f(x_3) \right]$$

$$= 2 \left[v(0) + v(2) + v(4) + v(6) \right]$$

$$= \boxed{\frac{1624}{585}} \approx \boxed{2.776068376}$$

approx